MEASURE AND INTEGRATION – FINAL EXAM
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1. (a) **[10 points]** Let Ω be a set and $\mathcal{E}$ be a collection of subsets of Ω. Assume that we have sets $A_0 \subset A \subset \Omega$ such that $A_0 \neq A$ and

   $$\text{for all } B \in \mathcal{E}, \quad \text{either } A \subset B \text{ or } A \cap B = \varnothing.$$

   Prove that $A_0 \notin \sigma(\mathcal{E})$, where $\sigma(\mathcal{E})$ denotes the σ-algebra generated by $\mathcal{E}$.

   (b) **[10 points]** Let Ω be a set and $\mu^\star$ be an outer measure on Ω. Suppose $A \subset \Omega$ is $\mu^\star$-measurable. Show that, for any $B \subset \Omega$ with $\mu^\star(B) < \infty$, we have

   $$\mu^\star(A \cup B) = \mu^\star(A) + \mu^\star(B) - \mu^\star(A \cap B).$$

2. **[15 points]** Let $E \subset \mathbb{R}$ be a Lebesgue measurable set (that is, $E \in \mathcal{M}$). Prove that, if $m(E) > 0$, then for every $\varepsilon > 0$ there exists an interval $[a, b]$ such that $m(E \cap [a, b]) > (1 - \varepsilon) \cdot (b - a)$.

3. In the following items, $\mathbb{R}$ and $\bar{\mathbb{R}}$ are endowed with the Borel σ-algebra.

   (a) **[10 points]** Let $(\Omega, \mathcal{A})$ be a measurable space, and let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. Define $f : \Omega \to \bar{\mathbb{R}}$ by

   $$f(\omega) = \inf\{n : \omega \in A_m \text{ for all } m \geq n\}, \quad \omega \in \Omega$$

   (we adopt the convention that $\inf \varnothing = \infty$). Prove that f is measurable.

   (b) **[10 points]** Let $f : \mathbb{R} \to \mathbb{R}$ be right continuous. Prove that f is measurable.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

   (a) **[7 points]** Let $f : \Omega \to \bar{\mathbb{R}}$ be an integrable function satisfying $\int_E f \, d\mu \geq 0$ for all $E \in \mathcal{A}$. Prove that $f \geq 0$ almost everywhere.

   (b) **[8 points]** Let $g : \Omega \to \bar{\mathbb{R}}$ be an integrable function. Prove that

   $$\lim_{n \to \infty} n \cdot \mu(\{\omega : g(\omega) > n\}) = 0.$$

5. In this exercise, we consider the set $\Omega = (0, \infty)$ with the Borel σ-algebra and Lebesgue measure (these are just the restrictions to $(0, \infty)$ of the Borel σ-algebra and Lebesgue measure of $\mathbb{R}$).
   Let $p > 1$ and let q be the conjugate exponent of p, that is, $p + q = pq$. Assume $f \in L^p((0, \infty))$.

   (a) **[6 points]** Show that, for every $x > 0$, $f \cdot \mathbb{1}_{(0,x)} \in L^1((0, \infty))$.

   (b) **[7 points]** Prove that, for any $\alpha \in (0, 1/q)$ and $x > 0$,

   $$\left| \int_0^x f(t) \, dt \right| \leq \frac{x^{\frac{1}{q} - \alpha}}{(1 - \alpha q)^{\frac{1}{q}}} \left(\int_0^x t^{\alpha p} \cdot |f(t)|^p \, dt \right)^{\frac{1}{p}}.$$

   Hint. Write $f(t) = t^{-\alpha} \cdot t^{\alpha} \cdot f(t)$ and use Hölder's inequality (make sure to verify that the assumptions for the inequality are satisfied!)

   (c) **[7 points]** Define, for $x > 0$,

   $$F(x) = \frac{1}{x} \int_0^x f(t) \, dt.$$

   Prove that $F \in L^p((0, \infty))$. *Hint.* You will need part (b) and the Fubini-Tonelli theorem. Use the fact that $p - \frac{p}{q} = 1$.